**2 Particle Interactions (Orbits)**

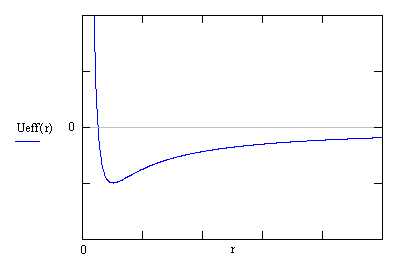
Let’s do a qualitative examination of orbits. We’ll go back to:



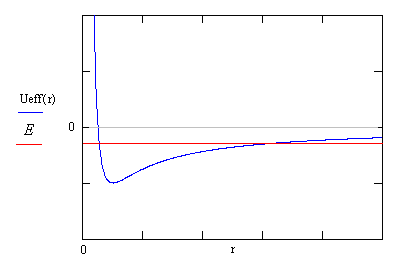
It is useful to combine angular kinetic energy and gravitational potential energy into one term and define Ueff.(r) = ℓ2/2μr2 + U(r), and to define the radial kinetic energy KEr = μvr2/2. Then we can write:



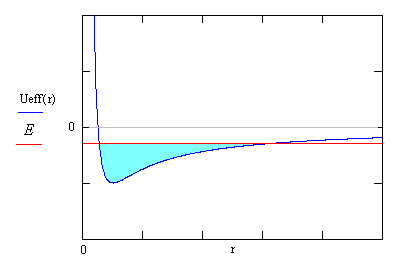
Now let’s display a typical plot of Ueff.(r), using U(r) = -k/r, though we could easily do this for any other potential.



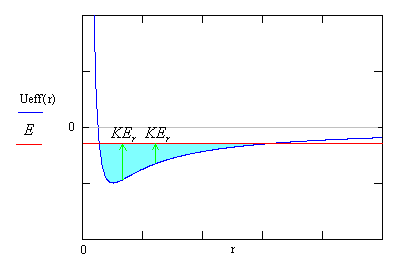
Generally, it goes to ∞ as r → 0, and then to 0 as r → ∞, as you can tell from the formula itself. Now the energy, , is a constant which must always be greater than or equal to Ueff(r), since vr2 must always be positive or 0. Let’s display a typical plot for then,



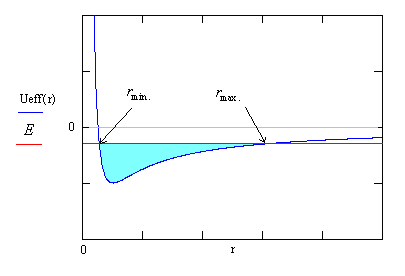
Now since the energy of the particle μ can only be greater than or equal to Ueff.(r), the particle can only exist in the region where the red line is above the blue curve, and this is highlighted below:



Outside this region is called the forbidden zone. The difference between  and U­eff.(r) is the radial kinetic energy. Two points in the path, and their respective radial KE values are shown below:



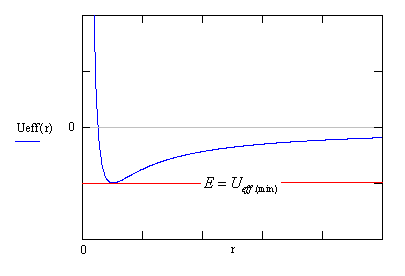
We will note that there are two special points located at the intersection of  and Ueff.(r). These are where KEr = 0, and hence are the so-called turning points, where the particle starts to turn back around and go in the other direction. These points would be the distance of furthest and closest approach.



Judging from the graph, there are 4 special cases.

**Case 1: Circular orbit**

This happens when  = minimum of Ueff., shown below.



Recalling,



we can find this minimum energy and the associated radius. We need to solve:



What is the value of the energy here? Well, since the radial potential energy is zero, it’s just Ueff(r0) itself. We’ll put this in terms of the radius r0.



So



which is just half the potential energy.



This is just what we get by applying N2L to circular orbits.



What’s the angular mometum? Well, from above, we have:



So,



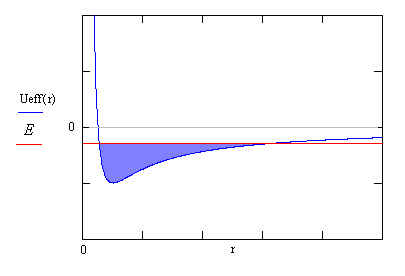
and kinetic energy? This is just KE = ℓ2/2μr2, since we have a circular orbit. So,



So we have KE = -(1/2)U.

**Case 2: Elliptical orbit**

In this case, we have Ueff.(min) < < 0. And in this case there are two allowed radii: rmin and rmax. This corresponds to an ellipse.



Let’s work out the turning points. Going back to our energy conservation equation, we have:



At the turning points, the radial velocity is zero. So then we have:



and so,



Using,



we come to:



What’s the sum of rmin and rmax, well, specifically, the average, which is the semimajor axis length, a = (rmin + rmax)/2?



What happens if we take the difference of the two (divided by 2)? Then:



We can take the ratio of the difference to the sum and get:



So,



That’s a cool result. And consider:



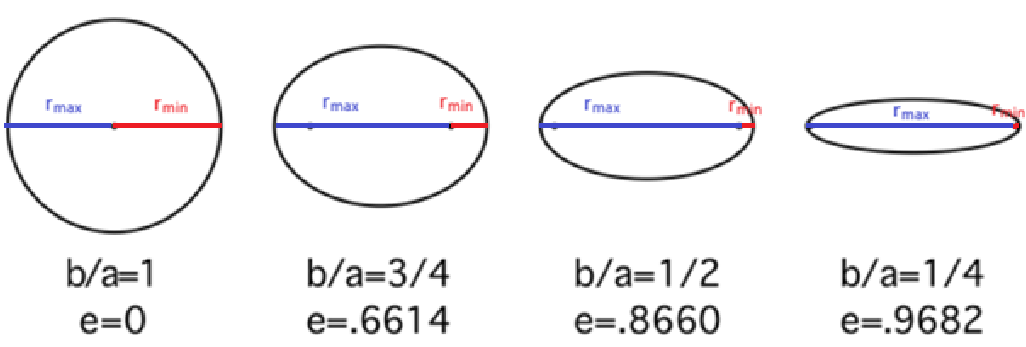
And we can use this to show that:



We also have, from standard ellipse stuff, that,



where b is the semiminor axis length, and f are the coordinates of the focii. Here’s a picture of a couple ellipses with same *a*, but different ε (b is the semiminor axis length).



Now let’s relate these to the dynamical quantities. So, recalling the explicit formula for the semimajor axis length up above:



we have:



So conversely, the energy is:



How about the potential energy? Well, the min (most negative) is given by:



Can we say something about the angular momentum? So the eccentricity is:



Let’s put ℓ in terms of ε,



Filling in ε in terms of the radii and such,



Could also say,



So,



Neat! For an orbit with fixed semimajor axis length *a*, which one has the largest angular momentum? I suppose it’s a circle? Yeah, looking back at the ℓ2 = kμa(1-ε2) formula. But we can also differentiate the present formula.



which would imply rmin = a too. So that is indeed for a circle. So the more eccentric the orbit, the less the angular momentum, for a given *a*. On the other hand, if I keep rmin fixed and just increase rmax, then ℓ2 should increase. I guess that makes sense. Let’s look at the kinetic energy. We’ll go back to:



So,



the maximum KE would be given by:



Suffice to say, the smaller rmin is, the larger KEmax is. Could also reason thus: at rmin (or rmax), the radial kinetic energy is zero, and so the kinetic energy is just the angular kinetic energy. And this is:



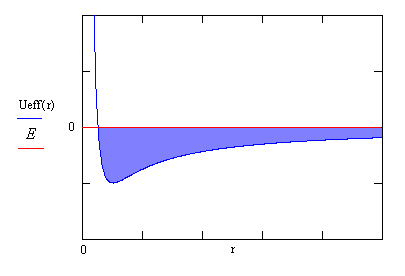
as well. So,



So as rmax increases, KEmax goes up, and as rmin increases, KEmax goes down.

**Case 3: Parabolic orbit**

This is when = 0, shown below. Such orbits we’ll find to be parabolic orbits. These orbits are not closed, and so the particle μ would make come by for a distance of closest approach but then ultimately fly off into space, never to come back.



Let’s work out the turning point. Going back to our energy conservation equation, we have:



At the turning point, the radial velocity is zero. So then we have:



and so,



Using,



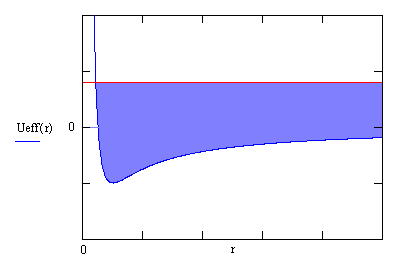
we come to:



Not sure how to generally get E, ℓ2, etc., from geometrical features of the orbit.

**Case 4: Hyperbolic orbit**

Similarly, we have case 4 which is when > 0. These orbits we’ll find to be hyperbolic. Again there would be a distance of closest approach, but ultimately they would fly away and never come back.



Again, we can work out the turning point. Let’s work out the turning point. Going back to our energy conservation equation, we have:



At the turning point, the radial velocity is zero. So then we have:



and so,



Using,



we come to:



Not sure how to generally get E, ℓ2, etc., from geometrical features of the orbit.

**Stable orbits**

Now we’d like to investigate how stable the closed orbits above are. We’d like to know, which central forces can support stable circular orbits? Basically, we need to know which attractive potentials will be such that the effective potential has a local minimum. So let our potential be:



Then the effective potential is:



It will have a local extremum where,



It will be stable when the 2nd derivative there is positive, i.e., when,



So we must have n < 2 in order to support a stable circular orbit.

**Example**

Suppose we locate an asteroid (m = 1.47×1015 kg) in the vicinity of the Sun (M = 2×1030 kg) a distance of r0 = 3×1011m away from the Sun’s center. And suppose its velocity is measured to be: **v**0 = 2.11×104m/s . Then what kind of orbit will it have, and what will be its distance of closest approach, furthest approach to the Sun?

Well first, we’ll calculate, and ℓ.



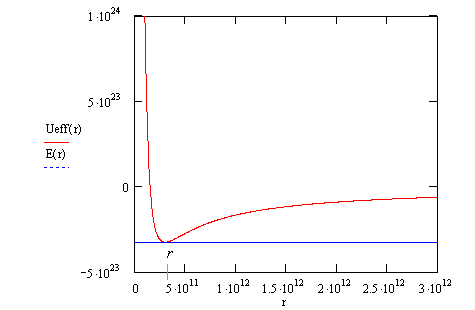
and ℓ is:



Now we’ll plot  and Ueff.(r), which is:



Obtaining,



**Example**

Suppose we give the asteroid velocity, **v**0 = 2.11×104 + 1.5×104 . Then what kind of orbit will it have, and what will be its distance of closest approach, furthest approach to the Sun?

To visualize the answer, let’s go back and recalculate , and ℓ.



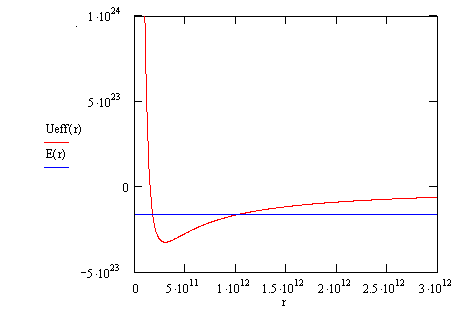
and ℓ is unchanged since we didn’t change vt or r0.



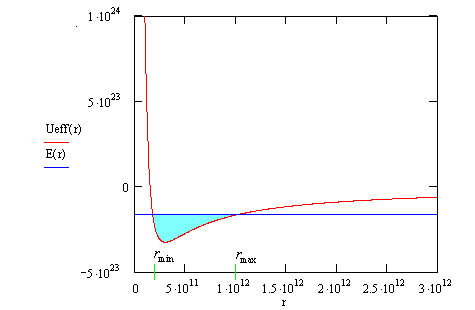
Now let’s plot, and Ueff.(r) which is also unchanged



as a function of position.



Now since must be greater than or equal to Ueff.(r), the only radii which the asteroid can have are the ones for which > Ueff.(r), i.e., the blue shaded region.



So this indicates that the asteroid will orbit the Sun with radii between rmin and rmax. Therefore, it will be orbiting elliptically. These radii can be determined numerically from a graphing calculator. They can also be solved for analytically. Observe that the min/max radii occur when KEr = 0. This means that,



and so,



Using,



we come to, as before:



**Example**

Now let’s take the same asteroid and boost its radial velocity even more, so that. **v**0 = 2.11×104 + 2.5×104 . Now what will its orbit look like?

Again, let’s calculate and ℓ.



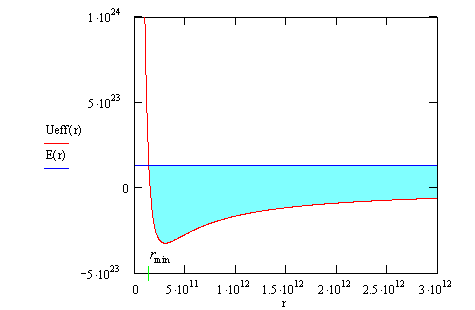
(note is positive) and ℓ is unchanged since we didn’t change vt or r0.



Now let’s plot and Ueff.(r) which is also unchanged



as a function of position, as well as r­­min and the permitted region.



Since > 0, there is no finite rmax. Therefore the orbit is unbounded. It is hyperbolic. The distance of closest approach can be obtained as above using the + sign (the – sign just gives a negative r value which doesn’t make any sense in the context).



We will get parabolic orbits when the energy is 0 and hyperbolic orbits for anything higher. There are some observations we should make. First, if is negative, then the orbit is closed. It will be either circular or elliptical. A circular orbit has the minimum possible energy. Elliptical orbits have more energy than circular ones. As the energy of the orbit increases, the eccentricity of the ellipse will increase more and more; that is rmin and rmax will become more and more separated. If becomes positive, the asteroid will have enough energy to break away from the Sun. Then the orbit is open, and parabolic.

**Example**

An Ariane 6 rocket launches into space from French Guiana with a payload of multiple small satellites. At an altitude of 800 km it injects the first satellite into orbit. The velocity of the satellite at that instant is 10 km/s, and the velocity makes an angle of 82o with the geocentric radius vector. Given that RE = 6.37×103 km and ME = 5.97 × 1024 kg, calculate the minimum height of the satellite above the earth's surface during its orbit.

So we’ll go back to:



We’re looking for the radial extrema, so set dr/dt = 0,



and solve for r,



Let’s just take μ = m,



The minimum radius is therefore,



Can write this as:



Now let’s get the initial energy, angular momentum, etc.,

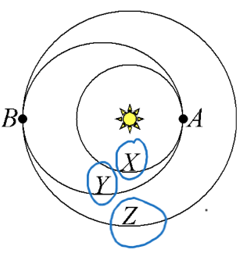
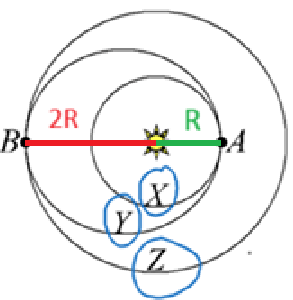


Filling all of these in,



**Example**

Rank the orbits in order of energy, min potential energy, angular momentum, and max kinetic energy. The largest orbit has radius 2R, while the smallest has radius R.

For energy, we have:



Generally, energy decreases with orbital diameter. For Umin, we have:



Generally, min potential energy will be predicated on how close the particle gets to the sun. For angular momentum, we have:



As can see from the general ℓ2 just above, ℓ2 increases with increasing rmax and rmin. So the larger these are, the larger ℓ2 is. And now for KEmax.



Maybe a qualititative way to think about it is, the tighter the orbit, the greater the KE, so KEmax,X > KEmax,Z for sure. But for orbit Y, we can reason that it must be greater than KEmax,X so as to be able to slingshot itself into higher orbit. On the other hand, a similar argument would seem to suggest that KEmax,Y should have less energy than KEmax,Z. So I don’t know. Might be more reliable just to calculate KEmax = E – Umin = -k/(2a) + k/rmin and compare.